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Regular uniform main-effect designs derivable from geometric factorial designs in 2^n runs

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Abstract. The article introduces a general method of construction of asymmetrical regular factorial main-effect designs in 2^n runs. It presents a collection of optimal designs constructed by this method in 32, 64, 128, and 256 runs. The method provides exploration of design structure and construction of designs with required properties. Construction of composite designs is given as an example of design structure exploration.

Keywords: Factorial design, geometric design, hypercube, regular factorial design, finite projective geometry, finite Euclidean space

1. Introduction

It is both pleasant and honorable for the authors of this paper to participate in this special issue in commemoration of Professor Sergey Aivazian and his substantial contribution to applied statistics. The next two paragraphs are the reminiscences about Sergey Aivazian by one of the authors, Slava Brodsky. These memories are about Sergey Aivazian's participation in organizing conferences on applied statistics, as well as about his work as a member of the editorial board of a journal section on applied statistics.

In the 70s and 80s, many people knew Sergey Aivazian from those conferences on applied statistical methods that he was conducting. Me and a number of my colleagues were their participants. Although the planning of the experiments is a preliminary stage for statistical analysis, I was among those who suggested that Sergey included works on the design of experiment into the program of his conferences. He approved that, and since then, his conferences have had included presentations on the design of experiment. I remember the 1979 conference in Tsakhkadzor. Aivazian was the head of the All-Union School of Applied Statistical Analysis. My colleagues and I made a presentation there (Brodsky et al., 1979). I remember that other scientists from various organizations also presented their works on the design of experiment at this conference. Of these, I remember well the participation of Valery Fedorov (who at that time was one of the leaders of that field of the applied statistics). At one point, Sergey and I played soccer against one of the local teams of the professional league. But that was only once as Sergey was usually playing tennis. Then there was another event in 1983. And works on the design of experiment were presented there too.

Sergey and I were not friends (I called him "Sergey Artemievich" – not Sergey, he called me "Vyacheslav Zinovievich" – not Slava), although we had known each other for a long time (I was then married to Tanya Golikova, and she and Sergey studied together at Moscow University). We closely communicated with him at the meetings of

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the mathematical section of the journal "Industrial Laboratory", where we both worked on a volunteer basis for about 25 years. And almost all my memories of Sergey Artemievich come from there. It was the only journal in the USSR that published articles on applied statistics. There was something about the nature of this section that distinguished it from other editorial boards: close connection between reviewers an authors of works. If an article contained a sound idea, then nothing – not a confusing explanation, not bad writing – could stop it from being published (albeit after revision). Sergey Aivazian was among of those who supported this high level work with the authors.

Now about the article. It presents methods of construction of regular uniform main-effect (RUME) designs in runs. These methods are based on the mathematical theory of symmetrical factorial designs, which was elaborated in Bose's (1947) and Rao's (1946, 1950) seminal papers as well as in theirs and other authors' subsequent works. Rao introduced a concept of hypercubes of strength t, which are designs consisting of finite Euclidean space points. Coordinates of each point satisfy a system of linear equations with certain properties. Brodsky (1976) brought into consideration more general constructions. He called such designs "geometric factorial", the name selection putting an emphasis on the method of construction rather than on the combinatorial features of the plans. That allows consideration of more general constructions retaining useful mathematical features.

The designs constructed in the article involve many number of observations and many variables. An area of application of our results, as we see it, is basically a screening procedure when the researcher is trying to determine significant variables among others (see recent developments described in works of Pojic et al. (2015) and Yurata et al. (2020) where the authors emphasize usefulness of factorial designs for screening).

Section 2 of the article contains necessary definitions and results that are used throughout the paper.

Section 3 considers two-level RUME-designs that are constructed from points of finite Euclidean space $EG(m, 2)$, coordinates of the points being consistent with a system of linear equations, or, equivalently, constructed from points of finite projective geometry $PG(n-1, 2)(m \leqslant 2^n)$. Such designs are called geometric. Then we construct asymmetrical RUME-designs (which are called generalized geometric) from two-level geometric RUME-designs. We introduce an origin matrix of the generalized geometric design that is instrumental in construction of the designs.

In Section 4, we investigate properties of the origin matrices for various particular cases. That allows to obtain new important results and construct new designs.

A summary of the results is given in Section 5.

2. Basic definitions and preliminary results

In this article, we consider symmetrical designs (that have equal numbers of levels for each factor) and derived from them asymmetrical designs (that have different numbers of levels). We follow definitions and results of theory of the factorial design of experiments developed originally by Brodsky (1976, 1983, 2013, 2019). Following him, we call a design uniform if every level of any factor occurs in the design equal for the given factor number of times.

Brodsky's next definition is based on a fundamental concept introduced by Plackett (1946) – the condition of proportional frequency.

Let $W_{1,2}^{j_1 j_2}$ be the number of simultaneous occurrences of the j_1-th level of the factor F_1 with the j_2-th level of the factor F_2; $W_1^{j_1}$ – the number of the j_1-th level of the factor F_1 occurrences; $W_2^{j_2}$ – the number of the j_2-th level of the factor F_2 occurrences; N – the number of runs. If proportional frequency condition

$$N W_{1,2}^{j_1 j_2} = W_1^{j_1} W_2^{j_2} \quad (1)$$

holds for two arbitrary factors F_1 and F_2, such design is called regular design of strength 2, or regular main-effect design.

For the uniform design, the condition Eq. (1) is equivalent to the following:

$$s_1 s_2 W_{1,2}^{j_1 j_2} = N, \quad (2)$$

where s_1 is the number of levels of the factor F_1, s_2 – the number of levels of the factor F_2.

Thus, the condition Eq. (2) determines regular uniform main-effect (RUME) designs. It is known (Brodsky, 1976) that RUME-design allows to get a diagonal covariance matrix of the design for the main effect model providing transformation from factor levels of the designs to regressional variables had been specially chosen. (For 2-level variables, for example, such transformation substitutes one level of factor with -1 and second level, with $+1$.) Moreover, RUME-designs are A-, D-, G, and Q-optimal for such models.

We will denote the design that includes the factors $F_1, F_2, \ldots, F_n$, each possessing $s_1, s_2, \ldots, s_n$ levels accordingly by

$$s_1 s_2 \ldots s_n // N.$$

When referencing factors with the equal number of levels, we will use a power sign. As appears from the condition Eq. (1), each RUME-design in 2^n runs may be represented in the following general form:

$$2^{n_2} 4^{n_4} 8^{n_8} \ldots // 2^n.$$

The nature of the frequency condition points to resemblance between RUME-designs and various combinatorial schemes. That has led to developing of a number of productive geometric methods aimed at constructing RUME-designs and based on the theory of finite Galois fields.

Let's now state basic facts that support the use of finite projective geometries tools for constructing RUME-designs in 2^n runs. Consider a set of all different combinations $(x_1, x_2, \ldots, x_n)$, where x_i $(i = 1, \ldots, n)$ are elements of Galois field *GF*(2), i.e. 0 or 1, and all x_i $(i = 1, \ldots, n)$ are not equal to zero simultaneously. Thus, the defined set forms a finite projective geometry *PG*$(n-1, 2)$. We shall call the combination $(x_1, x_2, \ldots, x_n)$ a point.

Denote points $(1, 0, \ldots, 0), (0, 1, \ldots, 0), (0, 0, \ldots, 1)$ by $1, 2, \ldots, n$ correspondingly, and select them as a basis of *PG*$(n-1, 2)$. Then every point p from *PG*$(n-1, 2)$ may be represented as

$$\boldsymbol{p} = a_1 \mathbf{1} + a_2 \mathbf{2} + \ldots + a_n \boldsymbol{n} \tag{3}$$

where $a_1, a_2, \ldots, a_n$ are also elements of *GF*(2). Denote point p in Eq. (3) by $\mathbf{1}^{a_1} \mathbf{2}^{a_2} \ldots \boldsymbol{n}^{a_n}$. We will drop the terms of the form $\boldsymbol{i}^0$ and substitute $\boldsymbol{i}$ for $\boldsymbol{i}^{\mathbf{1}}$. Let's put points $1, 2, \ldots, n$ into correspondence with $c_1, c_2, \ldots, c_n$-the 1-st, 2-nd, …, n-th columns of the standard complete $2^n // 2^n$ design. The standard form of the matrix of that design is

$$\left\| \begin{matrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ 1 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ 1 & 0 & \ldots & 1 \\ 0 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \end{matrix} \right\| \tag{4}$$

Then put point $\mathbf{1}^{a_1} \mathbf{2}^{a_2} \ldots \boldsymbol{n}^{a_n}$ into correspondence with the column $a_1 c_1 + a_2 c_2 + \ldots + a_n c_n$, where the addition is by modulo 2 and is performed row by row. For example, for $n = 5$, the sum of first, second and fifth columns of standard complete design 2^5// 32 corresponds to the point $\mathbf{125}$. In that way, *PG*$(n-1, 2)$ points will correspond to the columns of some $2^{(2^n - 1)} // 2^n$ design. It may be shown that this design is a geometric factorial design (GFD), and therefore it is a RUME-design (Brodsky, 2013). It follows from the replacement technique (Addelman, 1962; Brodsky, 1976) that any 2^{l-1} points of *PG*$(n-1, 2)$ that belongs to an arbitrary $(l-1)$-plane may be combined to form a 2^l-level factor. The simplest example of the replacement technique is the following correspondence between two 2-level factor and 4-level factor:

2-level factors		*4-level factor*
0 0		0
0 1	→	1
1 0		2
1 1		3

This correspondence means that we substitute columns of 2-level factors with one column of 4-level factor. We will call the resulting factor of the replacement procedure a reconstructed factor.

A design may include two "reconstructed" factors if and only if their forming planes do not intersect each other. We will call any RUME-design that includes reconstructed factors from a GFD a generalized geometric factorial design (generalized geometric RUME-design).

We do not know whether every RUME-design in 2^n runs is a generalized geometric RUME-design. It is possible to construct RUME-design that is not a GFD. An example of such $2^{15}//16$ design is:

$$\left\| \begin{matrix} \boldsymbol{D} & \boldsymbol{0} & \boldsymbol{D} \\ \boldsymbol{D}^* & \boldsymbol{1} & \boldsymbol{D}^{**} \end{matrix} \right\|,$$

where $\boldsymbol{D}$ is $2^7//$ 8 RUME design, $\boldsymbol{0}$ and $\boldsymbol{1}$ are correspondingly the 0-vector and 1-vector of 8 elements, design $\boldsymbol{D}^*$ is generated from design $\boldsymbol{D}$ by permutation of columns, design $\boldsymbol{D}^{**}$ is generated from design $\boldsymbol{D}^*$ by an inversion of levels 0 and 1. This design and GFD $2^{15}//$ 16 have different structure, in the sense that they cannot be converted one into another by permutation of rows and/or columns and by inversion of levels 0 and 1. On the other hand, we are unaware of existence of any RUME-design in 2^n runs containing at least one multilevel factor that does not belong to generalized geometric RUME-design class.

This paper presents methods of construction of generalized geometric RUME-design. All RUME-designs described in the literature as well as a number of new ones can be constructed using this method. Each design constructed in this way is called GFD or generalized geometric RUME-design and appears to be RUME-design.

3. Method of construction

Now we consider a method of construction of saturated designs, i.e. designs where a number of parameters to be estimated equals to a number of runs. Consider generalized geometric RUME-designs $R \times (2^s)^{h+1}//2^n$, where $R = 2^r$, $n = s + r$, $s \leqslant r$, $h \leqslant 2^{r-1}$ (r, s, h are positive integers). A lot of generalized geometric RUME-design of different types can be obtained out of these designs by splitting (Addelman, 1962; Brodsky, 1976) or by their structure investigation.

We will call matrix $\boldsymbol{M}^{r,s,h} = \{m_{ij}\}$ an origin matrix if

1) Elements of $\boldsymbol{M}^{r,s,h}$ are points of *PG*$(r-1,2)$;
2) Number of $\boldsymbol{M}^{r,s,h}$ rows is equal to $2^s - 1$;
3) Number of $\boldsymbol{M}^{r,s,h}$ columns is equal to h;
4) Any two elements of any row of $\boldsymbol{M}^{r,s,h}$ are different;
5) There exists such one-to-one correspondence between rows of $\boldsymbol{M}^{r,s,h}$ and points of *PG*$(n-1,2)$ that for any $k(1 \leqslant k \leqslant h)$, $t_1, t_2, t_3 (1 \leqslant t_1 < t_2 < t_3 \leqslant 2^{s-1})$ equality $f(t_1) = f(t_2) + f(t_3)$ is true if and only if $m_{t_1k} = m_{t_2k} + m_{t_3k}$, where $f(t_1)$, $f(t_2)$, $f(t_3)$ are points of *PG*$(s-1,2)$ corresponding t_1-th, t_2-th, t_3-th rows of matrix $\boldsymbol{M}^{r,s,h}$.

The requirement 5) means that the rows of matrix $\boldsymbol{M}^{r,s,h}$ are isomorphic to elements of *PG*$(s-1,2)$ in terms of addition. It follows from the origin matrix definition that $s \leqslant r, h \leqslant 2^{r-1}$.

Suppose that positive integers $s, r, h (s \leqslant r, h \leqslant 2^{r-1})$ are given. Now we choose arbitrary $(s-1)$-plane S in *PG*$(r-1,2)$ (which can be done because $s \leqslant r$) and set H consisting of h different points of *PG*$(r-1,2)$. Let's set the following isomorphism between *PG*$(r-1,2)$ and *GF*(2^r): point $\boldsymbol{p} = \boldsymbol{a}_1 1 + \boldsymbol{a}_2 2 + \ldots + \boldsymbol{a}_r r$ corresponds to point $2^0\boldsymbol{a}_1 + 2^1\boldsymbol{a}_2 + \ldots + 2^{r-1}\boldsymbol{a}_r$. So the matrix comprised of elements located at the intersection of the multiplication table rows, which correspond to points of S, and the table columns, which correspond to elements of H, is $\boldsymbol{M}^{r,s,h}$. So $s \leqslant r$ and $h \leqslant 2^{r-1}$ are necessary and sufficient for $\boldsymbol{M}^{r,s,h}$ existence.

The offered method is grounded upon the following

Statement 1. The $\boldsymbol{M}^{r,s,h}$ existence is equivalent to generalized geometric RUME-design $R \times (2^s)^{h+1}//2^n$ existence ($R = 2^r, n = s + r, s \leqslant r, h \leqslant 2^{r-1}, r, s, h$ are positive integers).

It follows from the above that it is necessary and sufficient to divide points of *PG*$(n-1,2)$ into nonintersecting $(r-1)$-plane (R-plane), which corresponds to 2^r-level factor, and $h+1$ $(s-1)$-planes, which correspond to 2^s-level factors, in order to construct such a design.

Choose R-plane generated by $(\boldsymbol{1}, \ldots, \boldsymbol{r})$ basis and some $(s-1)$-plane (S-plane) generated by $((\boldsymbol{r}+\boldsymbol{1}), \ldots, \boldsymbol{n})$ basis. The R-plane coincides with *PG*$(r-1, 2)$. It is obvious that these planes do not intersect each other. Since the basis of *PG*$(n-1, 2)$ is a combination of the basis of R- plane and the basis of S-plane, every *PG*$(n-1, 2)$ point belonging neither to R-plane nor to S-plane can be represented as the sum of a R-plane point and a S-plane point, and in a unique way. It allows us to represent points of other $(s-1)$-planes as the origin matrix $\boldsymbol{M}^{r,s,h}$.

Element m_{ij} corresponds to the point of j-th $(s-1)$-plane that is the sum of m_{ij} and i-th element of S-plane.

There exist a number of different origin matrices matching a given design. For instance, an origin matrix can be constructed by using any multiplication table of the *GF*(2^r) field as was described above.

In order to construct some design column, we are using linearly independent points of the corresponding plane. We will call them forming points. Each $(s-1)$-plane has s forming points while R-plane has r of them. A design column that corresponds to some factor can be reconstructed from columns of the complete design $2^n//2^n$ that corresponds forming points of the factor. Namely, the reconstructed factor is maintained at the i-th level if corresponding levels of factors of the complete design form the combination that is binary representation of i. For example, N-level factor is reconstructed according to following transformations from two-level factors:

$$\begin{array}{ccc} \textit{2-level factors} & & \textit{N-level factor} \\ 0\ 0\ 0\ \ldots\ 0\ 0 & & 0 \\ 0\ 0\ 0\ \ldots\ 0\ 1 & & 1 \\ 0\ 0\ 0\ \ldots\ 1\ 0 & \rightarrow & 2 \\ \ldots\ldots & & \ldots \\ 1\ 1\ 1\ \ldots\ 1\ 1 & & (N-1) \end{array}$$

We note here a few facts that considerably simplify construction of designs:

– The design column (let it be the first) that corresponds to S-plane consists of h zeros, h ones, ..., and h $(s-1)$ values;
– The design column that corresponds to R-level factor is $R//R$ design repeated s times;
– If any column of the origin matrix is equal to sum of other columns, the corresponding column of the design is equal to the sum of corresponding columns of the design and the first column.

Designs $8 \times 4^8// 32$, $16 \times 2^{16}// 32$, $16 \times 4^{16}// 64$, $32 \times 2^{32}// 64$, $16 \times 8^{16}// 128$, $32 \times 4^{32}// 128$, $64 \times 2^{64}// 128$, $64 \times 4^{64}// 256$, $32 \times 8^{32}// 256$, and $16^{17}// 256$ were first constructed by the offered method.

It is necessary to note that the Addelman-Kempthorne (1961) method with its generalization (Brodsky, 1981) allows to construct $2 \times 4^9//32$ and $2 \times 8^{17}//128$ designs that can be used to create $8 \times 4^8//32$ and $16 \times 8^{16}//128$ designs by appropriate replacement of two- and four-level factors.

The origin matrix $\boldsymbol{M}^{3,3,7}$, $\boldsymbol{M}^{4,3,15}$, $\boldsymbol{M}^{4,4,15}$ will be explored in the next section.

4. Origin matrix exploration

By origin matrix exploration, it is possible to obtain families of new generalized geometric RUME-design that cannot be obtained from $R \times (2^s)^{h+1}//2^n$ designs by usual transformations.

4.1. Origin matrix $\boldsymbol{M}^{\boldsymbol{3},\boldsymbol{3},\boldsymbol{7}}$ exploration

The origin matrix $\boldsymbol{M}^{\boldsymbol{3},\boldsymbol{3},\boldsymbol{7}}$ exploration allows to construct all possible generalized geometric RUME-design $8^k \times 4^m//64$. For $l < m$, generalized geometric RUME-design $8^k \times 4^m//64$ existence involves $8^k \times 4^l//64$ existence. So, the main problem here is the following: for every given *k,* to find the maximum value of m for which generalized geometric RUME-design $8^k \times 4^m//64$ exists.

Consider the origin matrix $\boldsymbol{M}^{\boldsymbol{3},\boldsymbol{3},\boldsymbol{7}}$ (the first row and first column do not belong to the matrix and are given for information purposes only):

R-plane contains 7 points of the form $\boldsymbol{1}^{a_1}\boldsymbol{2}^{a_2}\boldsymbol{3}^{a_3}$ while S-plane contains 7 points of the form $\boldsymbol{4}^{a_4}\boldsymbol{5}^{a_5}\boldsymbol{6}^{a_6}$. We remind that every origin matrix row corresponds to S-plane point. These points are adduced at the most left column.

The given matrix is matching generalized geometric RUME-design $8 \times 8^8//64$ (i.e. $8^9//64$) and, correspondingly, configuration of nine 2-planes in *PG*$(5, 2)$.

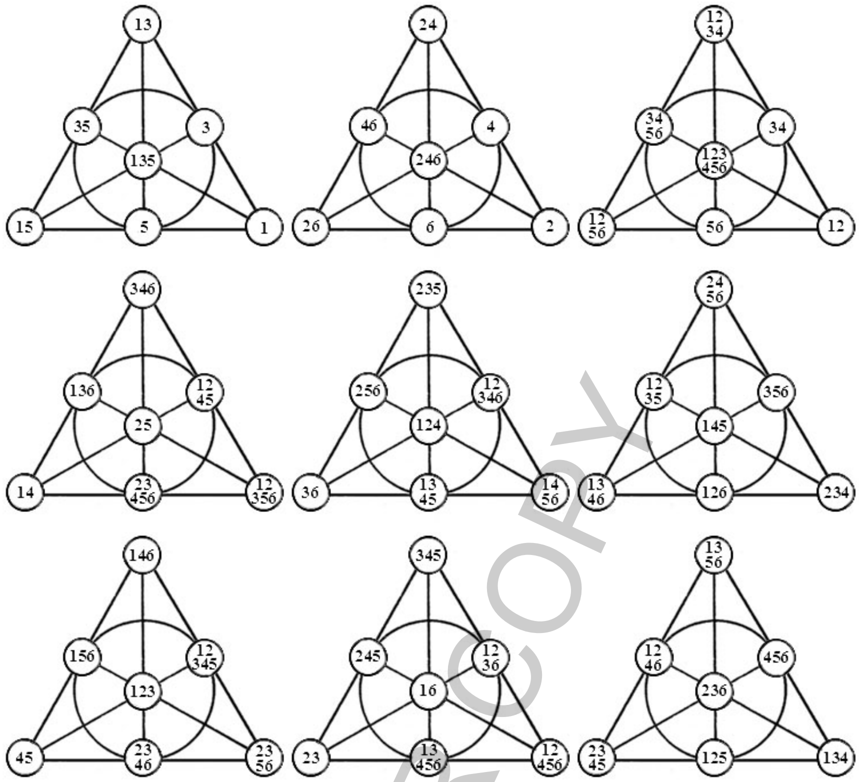

Fig. 1. A Magic Finite Projective Geometry *PG*$(5, 2)$.

	1	2	3	4	5	6	7
4	1	2	12	3	13	23	123
5	2	123	13	23	3	1	12
45	12	13	23	2	1	123	3
6	3	23	2	12	123	13	1
46	13	3	1	123	2	12	23
56	23	1	123	13	12	3	2
456	123	12	3	1	23	2	13

Consider an example (Brodsky, 1981) of constructing various asymmetrical main effect designs in 64 runs from the orthogonal arrays (64, 63, 2, 2). Consider each point of the complete design $2^{63}//64$ as point of the finite Euclidean space *EG*$(6, 2)$. There are 63 parallel pencils in *EG*$(6, 2)$ that form finite projective geometry *PG*$(5, 2)$. Put 63 factors of the orthogonal arrays (64, 63, 2, 2) into correspondence with the points of this geometry. Points $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_v \in PG(5, 2) (v \leqslant 6)$ are said to be linearly independent if

$$Rg\|\boldsymbol{x}_1, \ldots, \boldsymbol{x}_v\| = v.$$

Denote points $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1) \in PG(5, 2)$ by $1, 2, \ldots, 6$ respectively. These points are linearly independent. Any point $(a_1, \ldots, a_6) \in PG(5, 2)$ may by represented as a linear combination of points $1, 2, \ldots, 6$:

$$(a_1, \ldots, a_6) = \lambda_1(1, 0, \ldots, 0) + \ldots + \lambda_6(0, 0, \ldots, 1) \tag{5}$$

where $\lambda_i = 0, 1$ and all λ_i are not equal to zero simultaneously. Point Eq. (5) is denoted by $1^{\lambda_1}2^{\lambda_2} \ldots 6^{\lambda_6}$. E.g., point (1, 1, 1, 0, 1, 0) is denoted by 1235. All 63 points of *PG*$(5, 2)$ are presented in Fig. 1.

They are split into 9 subsets (displayed as triangles). Each subset contains 7 points. Only three of them are independent. Therefore, the seven points are located on a two-dimensional plane (2-flat). In any 2-flat, seven points are located on seven lines (1-flat). In Fig. 1, these lines passing through three points in each 2-flat are displayed as one circle and six line segments. For example, the left upper triangle in Fig. 1 represent a 2-flat containing seven points: 5, 35, 3, 135, 13, 15, 1. Points 1, 3, 5 can be chosen as linearly independent. The seven points are located on the seven lines: 15-35-13, 13-3-1, 15-5-1, 13-135-5, 15-135-3, 1-135-35, and 3-5-35.

The location of points of *PG*$(5,2)$ in Fig. 1 has the following additional property. Consider any three triangles (2-flats) located in the same row. Then the three points similarly located in the triangles belong to one so-called horizontal line (in 1-flat). E.g., in the three triangles of the second row, the central points 25,124, and145 belong to one line.

The construction in Fig. 1 resembles a magic square and therefore is called a magic finite projective geometry *PG*$(5,2)$.

Any three points located on the same line in *PG*$(5,2)$ belong to the parallel pencil corresponding to the main effects of the two factors and to their interaction effect. Therefore, to form one four-level factor we may use the replacement procedure. Similarly, any seven points located on one 2-flat may be used to form one eight-level factor.

Thus, each point in Fig. 1 is two-level factor, each line (1-flat) is four-level factor, each 2-flat is an eight-level factor.

This method produces the following four main effect designs: $4^{21}//64$ (21 lines), $8^3 \times 4^{14}//64$ (three 2-flat and 14 horizontal line), $8^8 \times 4^7//64$ (six 2-flats and seven horizontal lines), and $8^9//64$ (nine 2-flat).

Various designs of the form $8^m \times 4^n \times 2^l//64$ can be constructed in a similar way. However, any one of them can be derived from $8^9//64$, $8^6 \times 4^7//64$, or $8^3 \times 4^{14}//64$ by splitting procedure.

Consider the following construction:

246	1246	1
134	234	12
35	15	13
1256	356	123
1236	136	2
145	12345	23
23456	2456	3

13456	1356	4
16	1456	45
256	245	46
2346	235	456
345	34	5
125	126	56
1234	12346	6

1235	2345	14
346	12356	1245
236	124	1346
1345	26	123456
12456	146	25
24	3456	2356
156	135	36

Here, nine 2-planes are arranged in three rows and three columns. Three points from any of horizontal rows form 1-plane (line). These lines will be called horizontal. Three points, identically placed in three 2-planes of any column (for example, points 136, 34, and 146 in 2-planes of the second column) form a line, too. These lines will be called vertical.

By using this arrangement, we construct the following RUME-designs:

$8^9 \times 4^0//64,$

$8^8 \times 4^1//64,$

$8^7 \times 4^2//64,$

$8^6 \times 4^7//64,$

$8^5 \times 4^8//64,$

$$8^4 \times 4^{10}//64,$$
$$8^3 \times 4^{14}//64,$$
$$8^2 \times 4^{15}//64,$$
$$8^1 \times 4^{17}//64,$$
$$8^0 \times 4^{21}//64.$$

The RUME-design $8^8 \times 4^1//64$ corresponds to eight 2-planes (excluding 2-plane 1-12-13-123-2-23-3) and line 1-2-12.

The RUME-design $8^7 \times 4^4//64$ corresponds to seven 2-planes (excluding 2-planes 1-12-13-123-2-23-3 and 4-45-46-456-5-56-6) and 2 lines: 1-2-12 and 4-5-45.

The RUME-design $8^6 \times 4^7//64$ corresponds to six 2-planes of the second and third rows and 7 horizontal lines (drawn up from 2-planes of the first row points).

The RUME-design $8^5 \times 4^8//64$ corresponds to five 2-planes of the second and third rows (excluding 2-plane 4-45-46-456-5-56-6), 7 horizontal lines (drawn up from 2-planes of the first row points), and also 4-5-45 line.

The RUME-design $8^4 \times 4^{10}//64$ corresponds to four 2-planes (three 2-planes from the third column and also 1246-234-15-356-136 - 12345-2456 2-plane) and ten following lines:

- Six vertical lines 1236-345-12456, 246-13456-1235, 134-16-346, 145-125-24, 35-256-236, and 23456-1234-156 drawn up from 2-planes of the first column points;
- Line 1456-235-12346 drawn up from 2-plane of the points of the 2-nd row and the 2-nd column;
- Line 2345-124-135 drawn up from 2-plane of the points of the 3-rd row and the 2-nd column;
- Two lines: 2346-34-26 and 1256-245-146.

The RUME-design $8^3 \times 4^{14}//64$ corresponds to three 2-planes of the 3-rd row and 14 horizontal lines (drawn up from the 2-plane points of the 1-st and the 2-nd rows).

The RUME-design $8^2 \times 4^{15}//64$ corresponds to two 2-planes 1-12-13-123-2-23-3 and 1246-234-15-356-136-12345-2456, fourteen horizontal lines (drawn up from 2-plane points of the 1-st and the 2-nd rows) and also line 246-35-23456 (drawn up from 2-plane points of the 1-st row and the 1-st column).

The RUME-design $8^1 \times 4^{17}//64$ corresponds to 2-plane 1-12-13-123-2-23-3, four vertical lines 134-16-346, 1256-2346-1345, 234-1456-12356, 356-235-26, seven horizontal lines 345-34-5, 125-126-56, 1235-2345-14, 236-124-1346, 12456-146-25, 24-3456-2356, 156-135-36, five lines 4-46-6, 1246-15-2456, 1356-245-12346, 246-35-23456, 13456-256-1234, each of them drawn up from corresponding 2-plane points, and also 1236-45-123456 line.

The RUME-design $4^{21}//64$ corresponds to twenty one horizontal lines drawn up from all nine 2-planes.

It is easy enough to show that generalized geometric RUME-designs

$$8^k \,//\, 64 \qquad (k > 9),$$
$$8^8 \times 4^2 \,//\, 64,$$
$$8^6 \times 4^8 \,//\, 64,$$
$$8^5 \times 4^9 \,//\, 64,$$
$$8^4 \times 4^{11} \,//\, 64,$$
$$8^3 \times 4^{15} \,//\, 64,$$
$$8^2 \times 4^{16} \,//\, 64,$$
$$8^1 \times 4^{18} \,//\, 64,$$
$$4^k \,//\, 64 \qquad (k > 21)$$

do not exist.

The generalized geometric RUME-design $8^7 \times 4^3//64$ does not exist. The proof of that is rather complicated. It rests upon the following statement (Boguslavsky, 1989).

Statement 2. The origin matrix $\boldsymbol{M}^{3,3,k}$ for $k \neq 3$, $k \neq 7$ can be augmented with one column so that the resulting matrix would be $\boldsymbol{M}^{3,3,k+1}$.

Thus, for any k, m, the problem of construction of generalized geometric RUME-design $8^k \times 4^m//64$ is completely resolved.

4.2. *Origin matrix* $\boldsymbol{M}^{\mathbf{4,3,15}}$ *exploration*

Consider the following origin matrix of generalized geometric RUME-design $16 \times 8^{16}//128$:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	1	2	12	3	4	34	13	24	1234	23	124	134	14	234	123
6	2	12	1	4	34	3	24	1234	13	124	134	23	234	123	4
56	12	1	2	34	3	4	1234	13	24	134	23	124	123	14	234
7	3	4	34	14	234	123	134	23	124	1	2	12	24	1234	13
57	13	24	1234	134	23	124	4	34	3	123	14	234	12	1	2
67	23	124	134	1	2	12	123	14	234	24	1234	13	3	2	34
567	123	14	234	13	24	1234	2	12	1	34	3	4	134	23	124

R-plane contains 15 points of the form $\mathbf{1}^{a_1}\mathbf{2}^{a_2}\mathbf{3}^{a_3}\mathbf{4}^{a_4}$ while S-plane contains 7 points of the form $\mathbf{5}^{a_5}\mathbf{6}^{a_6}\mathbf{7}^{a_7}$. Let's split R-plane points into 5 mutually nonintersecting lines: $L1$ (34-13-14), $L2$ (123-134-24), $L3$ (124-4-12), $L4$ (1-234-1234), $L5$ (2-3-23). Every line corresponds to 4-level factor. Then split $\boldsymbol{M}^{\mathbf{4,3,15}}$ columns into 5 groups 3 columns each: (1, 2, 3), (4, 5, 6), (7, 8, 9), (10, 11, 12), and (13, 14, 15). Denote the points of the groups by $a[i]$, where a is an element of S-plane and $i = 1, 2, 3$. In other words, points $a[i]$ of the finite geometry correspond to elements the origin matrix that belong to the a-th row and i-th column of any group of columns.

For 3 first groups, lines consist of points 6[1]-56[2]-5[3], 5[1]-6[2]-56[3], 56[1]-57[1]-67[1], 5[2]-67[2]-567[2], 6[3]-7[3]-67[3], 7[1]-7[2]-*, 57[2]-57[3]-*, 567[1]-567[3]-*, where asterisk denotes necessary point of $L1, L2, L3$. For the 4-th group, lines consist of points 523-6134-56124, 6124-5623-5134, 71-56134-56734, 5124-671234-5673, 623-57234-5674, 6724-6713-1234, 57123-5714-234, and 72-712-1. There exist similar splitting for the points of the 5-th group and $L5$.

Each group sets 21 points in *PG*$(6, 2)$. These points together with $L1, L2, L3, L4, L5$ points correspondingly can be split into 8 pairwise nonintersecting lines. By doing these substitutions consecutively we can construct 3 new saturated generalized geometric RUME-design:

$$8^{13} \times 4^{12}//128, 8^{10} \times 4^{19}//128, \text{ and } 8^7 \times 4^{26}//128.$$

4.3. *Origin matrix* $\boldsymbol{M}^{\mathbf{4,4,15}}$ *exploration*

Consider the following origin matrix of generalized geometric RUME-design:

R-plane contains 15 points of the form $\mathbf{1}^{a_1}\mathbf{2}^{a_2}\mathbf{3}^{a_3}\mathbf{4}^{a_4}$, while S-plane contains 7 points of the form $\mathbf{5}^{a_5}\mathbf{6}^{a_6}\mathbf{7}^{a_7}\mathbf{8}^{\boldsymbol{a_8}}$.

Now we divide 15 columns of the matrix into 5 groups of 3 columns each: (1, 2, 3), (4, 5, 6), (7, 8, 9), (10, 11, 12), and (13, 14, 15). Here we will use the same notations as introduced before. Each group of columns sets 45 points in *PG*$(7, 2)$ that can be split into pairwise nonintersecting three 2-planes and eight lines. The 2-planes consist of points 7[1]-8[1]-78[1]-5[1]-57[1]-58[1]-578[1], 7[2]-8[2]-78[2]-6[2]-67[2]-68[2]-678[2], and 7[3]-8[3]-78[3]-56[3]-567[3]-568[3]-5678[3]. The lines consist of points 6[1]-56[2]-5[3], 56[1]-5[2]-6[3], 68[1]-5678[2]-57[3], 5678[1]-57[2]-68[3], 67[1]-568[2]-578[3], 568[1]-578[2]-67[3], 567[1]-58[2]-678[3], and 678[1]-567[2]-58[3].

By substituting three 8-level and eight 4-level factors for three 16-level factors of the GGFD $16^{17}//256$ consecutively, according to the splitting method described above, we constructed new GGFDs: $16^{14} \times 8^3 \times 4^8//256$, $16^{11} \times 8^6 \times 4^{16}//256$, $16^8 \times 8^9 \times 4^{824}//256$, $16^5 \times 8^{12} \times 4^{32}//256$, and $16^2 \times 8^{15} \times 4^{40}//256$ that cannot be constructed by known transformations.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	1	2	12	3	4	34	13	24	1234	23	124	134	14	234	123
6	2	12	1	4	34	3	24	1234	13	124	134	23	234	123	14
56	12	1	2	34	3	4	1234	13	24	134	23	124	123	14	234
7	3	4	34	14	234	123	134	23	124	1	2	12	24	1234	13
8	4	34	3	234	123	14	23	124	134	2	12	1	1234	13	24
78	34	3	4	123	14	234	124	134	23	12	1	2	13	24	1234
57	13	24	1234	134	23	124	4	34	3	123	14	234	12	1	2
68	24	1234	13	23	124	134	34	3	4	14	234	123	1	2	12
5678	1234	13	24	124	134	23	3	4	34	234	123	14	2	12	1
58	14	234	123	24	1234	13	12	1	2	3	4	34	23	124	134
678	234	123	14	1234	13	24	1	2	12	4	34	3	124	134	23
567	123	14	234	13	24	1234	2	12	1	34	3	4	134	23	124
67	23	124	134	1	2	12	123	14	234	24	1234	13	3	4	34
568	124	134	23	2	12	1	14	234	123	1234	13	24	4	34	3
578	134	23	124	12	1	2	234	123	14	13	24	1234	34	3	4

4.4. Composite designs in 2^n runs

Compositeness is very important property of designs that researchers use in practice. After the experiments have been carried out, the response surface often proves to have more complicated form than it was assumed when the experiments were originally planned. So, it may be necessary to conduct an additional series of experiments. In this situation, keeping already carried out experiments inside of the new design is a natural desire. A design that includes the original series of experiments as a part of second series of experiments is called composite.

Construction of a composite design for $R \times s^R//2^n$ generalized geometric RUME-design (where $R = 2^r, S = 2^s, n = s+r, s \leqslant r, s$ and r are positive integers) can be performed from $R \times S^{h+1}//2^n$ generalized geometric RUME-design when $h = 2^{r-1}$. It will consist in searching for such trials of $R \times S^R//2^n$ design that coincide with those of the design $R \times (S/2)^R//2^{n-1}$. It should be noted that generalized geometric RUME-design $R \times (S/2)^R//2^{n-1}$ contains the levels of just $(S/2)$-level factors.

Let's take the $R \times S^R//2^n$ design and consider a trial in which a particular S-level factor ($S = 2^s$) has a value below $S/2$. At that trial, the s-th forming point of the factor must be at zero level. Therefore, the basis factors of the $2^n//2^n$ design participating in the s-th forming point must be at zero level as well. It can be shown that for trials in which all the S-level factors have values below $S/2$, all of the basis factors with numbers from s to n must be at zero level. There are 2^{s-1} such trials. Their numbers are $1 + i2^r, i = 0, \ldots, 2^{s-1} - 1$. The more general statement is also true:

Statement 3. For $R \times S^R//2^n$ generalized geometric RUME-design (where $R = 2^r, S = 2^s, n = s + r, s \leqslant r, s$ and r are positive integers) in every row of the form $1 + i2^r (i = 0, \ldots, 2^{s-1})$ all S-level factors are maintained at level i while in any other row 2^{r-s} S-level factors are maintained at level 0, 2^{r-s} S-level factors are maintained at level $1, \ldots, 2^{r-s}$ S-level factors are maintained at level $(S - 1)$.

Thus, when going from the original design $R \times (S/2)^R//2^{n-1}$ to the second design $R \times s^R//2^n$, only $S/2$ runs remain in common. That is not enough for practical use. However, there may be found other composite designs for more complicated models. These composite designs keep all runs of the original designs.

We will illustrate that by a simple example. Consider $4^5//16$ design. Five 4-level factors correspond to the following points: 1-2-12, 3-4-34, 13-24-1234, 14-234-123, and 134-23-124. We will select points with notations that do not include "4" and get the $4 \times 2^4//8$ design. Then assume that in addition to the original five factors, the investigated response surface is influenced by 6-th factor that was maintained at some constant level in the original series of experiments. Then we can run another series of experiments with the additional factor maintained at its second level, and all the other 5 factors being at the same levels as in the original series. That allow us to estimate effects of the 5 original factors as well as the effect of the additional 6-th factor. Besides, we can estimate effects of interactions of the original factors with the additional one. For instance, 6-th factor may be a source of raw materials. The investigated response surface might be influenced by that. Combined design allows to estimate effect of that additional factor. Moreover, there might be interactions between the 6-th additional factor and some of the original factors. These effects can be estimated in the combined design as well. Note that any traditional composite design allows to estimate more complicated model for the same factors involved. The introduced composite designs allow estimating model with additional factors. We will call such designs A-composite (meaning that additional factors are involved in the investigation).

Any $R \times S^R//2^n$ design for $r \geqslant s$ allows to construct the A-composite design $R \times (2S)^R//2^{n+1}$ using a procedure similar to that described above. To achieve that, some $(2S)$-level factors of the design $R \times (2S)^R//2^{n+1}$ should be split into original and additional factors. After that, the original factors correspond to the forming PG points with notation that does not include point with number $s+1(2S = 2^{s+1})$. If it was done, half of $R \times (2S)^R//2^{n+1}$ design runs would form $R \times S^R//2^n$ design for original factors. During carrying out these experiments the levels of additional factors are to be planned in order the design might be augmented to $R \times (2S)^R//2^{n+1}$ design.

5. Conclusion

In the paper, we present a method of construction of geometric factorial designs $R \times S^{h+1}//2^{s+r}$, where $R = 2^r, S = 2^s, n = s + r, s \leqslant r, h \leqslant 2^{r-1}$: r, sh are positive integers. The method is based on an origin matrix $\boldsymbol{M^{r,s,h}}$ construction that consists of 2^{s-1} rows and h columns. The origin matrix can be constructed from $GF(2^r)$ multiplication table by setting R-plane points into correspondence with nonzero $GF(2^r)$ elements. In addition to construction of such designs, the method provides exploration of design structure and allows construction of A-composite designs.

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